

Klebanov-Witten theory with massive dynamical flavors

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ABSTRACT: We consider the addition of a large number of massive dynamical flavors to the Klebanov-Witten theory, the quiver gauge theory describing the low energy dynamics of N_c D3-branes at the conifold singularity. Massive flavors are introduced by means of N_f D7-branes which are holomorphically embedded and smeared along the transverse directions. After some general comments on the validity of the smearing procedure, we find the full backreacted supergravity solution corresponding to a particular class of massive embeddings. The solution depends on a running effective number of flavors, whose functional form follows from the smeared embedding. The running reflects the integrating in/out of massive degrees of freedom in the dual field theory as the energy scale is changed. We study how the dynamics of the theory depends on the flavor parameters, mainly focusing on the static quark-antiquark potential. As expected, we find that the dynamical flavors tend to screen the static color charges.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence.

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1. Introduction

The construction of string duals of gauge theories with dynamical flavors is a task of obvious interest. The gauge theories usually describe the low energy dynamics at the intersection of N_c “color” and N_f “flavor” D-branes and evaluating the backreaction of the full system is

not easy in general. If the flavor branes are on top of each other, the supergravity equations of motion cannot be simply reduced to ordinary first order differential equations in a radial variable. Instead, they are partial differential equations in a number of variables given by the number of directions transverse to both the color and the flavor branes. For D3D7 models, this kind of scenarios have been studied in [1]. Other localized constructions were discussed in [2].

To avoid this complication one can consider simplified set-ups where the flavor branes are homogeneously smeared [3] in the transverse space. Whereas this restricts the possible ways of introducing fundamental matter (one requires that the symmetries of the unflavored theory should be effectively unbroken), it gives a useful framework to analyze physical features of the resulting flavored theories. Moreover, the “smearing trick” is not only computationally helpful. As we are going to argue in the following section, it is in a sense a preferred choice if one wants to avoid singularities and large string couplings near the position of localized sets of branes.

Following this prescription, the construction of string duals of flavored supersymmetric theories in the Veneziano limit $N_c, N_f \gg 1$ with N_f/N_c fixed, has been possible. In [4–6] (resp. [7]) the string dual of a SQCD-like theory with massless (resp. massive) dynamical flavors was found. The setup is determined by D5-branes wrapped on compact (for the color branes) or non compact (for the flavor branes) two-cycles. The theory is the flavored version of the confining Chamseddine-Volkov-Maldacena-Nunez (CVMN) solution [8]. In [9] a large number of massless flavors was added to the conifold conformal theory of Klebanov and Witten (KW) [10]. In [11] the construction was extended to the confining Klebanov-Strassler [12] case. Other related setups and studies can be found in [13–16].

In this paper we focus on the flavored Klebanov-Witten model and extend the analysis of [9] by considering the case where the dynamical flavors are massive and all with the same constituent mass m_q . Fundamental flavor multiplets are added to the theory by means of D7-branes, which are wrapped on non-compact 4-cycles and holomorphically embedded in the background in order to preserve the $\mathcal{N} = 1$ supersymmetry [17]. The authors of [9] found the supergravity solution generated by a “massless”, smeared D3D7 system. This has a running dilaton and non trivial F_1 (sourced by the D7-branes) and F_5 RR fluxes. The near horizon geometry has no AdS factor and this translates in a broken conformal invariance in the dual field theory. The running dilaton blows up at a certain value of the radial coordinate corresponding to a field theory UV Landau pole. Moreover, the background has a curvature singularity when the dual gauge theory is in the far IR.

In the following sections we will show how with massive flavors one can find a IR regular solution. Of course, the latter will not alter the UV behavior of the field theory and the Landau pole will still be present. The flavor D7-branes are embedded in such a way that they reach a minimal radial distance ρ_q from the bottom of the space. This distance is related to the mass parameter m_q , just as the field theory energy scale is related to the radius ρ . Energies larger (smaller) than m_q map to radii larger (smaller) than ρ_q . The knowledge of the conifold geometry allows to find the density distribution of the smeared flavor branes as a function of the radial coordinate.

With the density distribution explicitly calculated, we are able to solve the supergravity

equations of motion coupled to the N_f D7-brane sources. The net result is interesting. As it was anticipated in [9], the first order equations for the background fields following from the supersymmetric fermionic variations retain the same form as those in the massless case, modulo a substitution of N_f with a running effective number of flavors, $N_f(\rho)$. The function $N_f(\rho)$ is related to the flavor density distribution mentioned above. This effective running of the number of flavors has a nice field theory interpretation. As we go towards energy scales much larger than the mass m_q the theory resembles the massless one; at energies lower than m_q , the flavors can be integrated out and the theory looks like the unflavored one. We will see that the function $N_f(\rho)$, as well as the whole supergravity solution we will deduce from it, precisely reproduces this field theory expectation. The function $N_f(\rho)$ in fact has the shape of a smoothed out Heaviside step function $N_f\Theta(\rho-\rho_q)$. In contexts where extracting the function $N_f(\rho)$ from the smeared massive embedding could be technically difficult, it is useful to plug in the supergravity equations just the simple Heaviside function. In this paper we will compare the results obtained by using the correct $N_f(\rho)$ with those deduced using the step approximation, finding that, in the flavored KW model at hand, the latter works quite well, at least at small m_q . This approximation was used to study novel effects of massive dynamical flavors in the confining CVMN background in [7].

Once the full flavored KW background has been found, one can use it to study how the strongly coupled dynamics of the dual gauge theory is affected by the dynamical flavors. One of the expected effects is the screening of the color charge. In this paper we use standard holographic techniques to extract the static potential between an external (i.e. extremely massive) quark-antiquark pair \bar{Q}, Q probing our flavored gauge theory. The potential results to have a Coulomb-like shape and its behavior as a function of the sea quark parameters N_f, m_q precisely accounts for the expected screening of the color charges. We also study the behavior of the screening length as well as that of the minimal quark-antiquark distance at which the $\bar{Q}Q$ bound state can decay into a pair of specific heavy-light bound states $\bar{Q}q + \bar{q}Q$ (by popping out from the vacuum a dynamical quark-antiquark pair \bar{q}, q). Our analysis shows that the screening length is an increasing (resp. decreasing) function of m_q (resp. N_f). The above mentioned minimal distance, which we call “string breaking distance” L_{sb} , is instead a decreasing function of both N_f and m_q .

The paper is organized as follows. We start in section 2 by reviewing the smearing technique, pointing out which are its limits of validity and its relevance. In section 3 we consider certain generalized massless and massive D7 embeddings on the conifold. Their smearing is considered in section 4 where the expression for their radial distribution density in a massive case is calculated. In section 5 we find the supergravity background dual to the KW theory coupled to massive dynamical quarks and discuss the regimes where we can trust the solution. In section 6 we rewrite the supergravity solution by using the Heaviside step function approximation. We then study, in section 7, the dependence on the flavor parameters of the static quark-antiquark potential and of the screening and string breaking lengths, making a comparison between the results found using the “true” supergravity solution and those following from the step function approximation. In appendix A we review the orbifold origin of some D7 embeddings on the conifold. In appendix B we study the dependence of the static quark-antiquark potential on a particular integration constant.

2. Comments on the smearing technique

The technical trick we adopt to construct the string duals of gauge theories with a large number of dynamical flavors, consists in considering the flavor branes as homogeneously smeared along the transverse directions [3]. This trick allows one to find relatively simple solutions having the largest possible degree of symmetry and taking into account the full backreaction due to N_c “color” and N_f “flavor” D-branes. Moreover, as we are going to argue, the trick is in a sense forced by the approximations used to find the full solution. In fact, the starting point on the low energy string side is an action of the form [18]

$$S = S_{\text{II}} + \sum_{i=1}^{N_f} [S_{i\text{BI}} + S_{i\text{WZ}}], \tag{2.1}$$

where the first term is the (IIB or IIA) bulk supergravity action and the remaining ones are the Dirac-Born-Infeld and Wess-Zumino part of the flavor brane action, with the flavor branes embedded in the background. One usually first solves for the embedding equations and then for the remaining bulk equations on-shell.

Let us now consider N_f *coincident* flavor branes, such that the corresponding flavor symmetry group is $U(N_f)$. In this case one expects corrections to the DBI action coming from the fact that an open string can end on the N_f branes, so that the effective coupling is $g_s N_f$. In the quenched approximation $N_f \ll N_c$ and the coupling is small, but if $N_f = O(N_c)$ the string theory would be strongly coupled. The DBI action was shown to appropriately describe the $g_s N_f$ -leading order dynamics of open strings in a generic background [19]. Thus, if $g_s N_f$ is order one, the corrections to the DBI can be large. This fact has a precise analogue in field theory in the Veneziano limit [20]. At any given order in N_c , the insertion of n quark loops (“windows”) is governed by the parameter $(g_{\text{YM}}^2 N_f)^n$ and can be done perturbatively only for small $g_{\text{YM}}^2 N_f$. If the latter is not small, the sum over any number of windows must be performed non-perturbatively. In the brane language, the “one window graph” corresponds to the DBI contribution, which is then a good approximation only for small $g_s N_f$.

Crucially, *in the smeared setup* the effective coupling $g_s N_f$ is further suppressed.¹ Due to the smearing, the flavor symmetry $U(N_f)$ is generically broken to $U(1)^{N_f}$ and the distance between two generic flavor branes is large (in string units) in the supergravity approximation. Now, a typical string process will involve a space-time region of size of order one in $\sqrt{\alpha'}$ units: in such a region, thus, only a small fraction of the N_f smeared flavor branes will be available for the process. If we denote as R the typical radius of an internal dimension of the geometry in $\sqrt{\alpha'}$ units, the number of flavor branes involved in a typical process will be of order N_f/R^d , where d is the codimension of the flavor branes (the number of dimensions involved in the smearing). Since in the supergravity approximation $R \gg 1$, the effective coupling to the flavor branes $(g_s N_f)/R^d$ will typically be small even if $N_f = O(N_c)$. Thus, in cases as [4]–[7, 13, 15] where the supergravity regime can be

¹We are grateful to Carlos Núñez for this crucial observation and for his relevant contributions to the discussion presented in this section.

attained with no restrictions on N_f/N_c , the use of the DBI action is still justified also for $g_s N_f$ of order one. For the case considered in this paper, nevertheless, we will show in section 5.5 that the validity of the supergravity approximation implies $N_f \ll N_c$ and that the validity of the DBI does not impose any further restriction.

A more detailed discussion of these issues will be presented in [21].

3. D7 embeddings on the conifold

The low energy dynamics of N_c D3-branes at the conifold singularity

$$z_1 z_2 = z_3 z_4, \tag{3.1}$$

where the z_i are complex coordinates, is described by an $\mathcal{N} = 1$ superconformal quiver gauge theory with gauge group $SU(N_c) \times SU(N_c)$ and bifundamental matter fields A_1, A_2 and B_1, B_2 transforming respectively in the (N_c, \bar{N}_c) and in the (\bar{N}_c, N_c) representations of the gauge group [10]. The matter fields form two $SU(2)$ doublets and interact through a quartic superpotential

$$W_{\text{KW}} = \epsilon^{ij} \epsilon^{kl} [A_i B_k A_j B_l]. \tag{3.2}$$

Here and in the following, traces over color indices are implied. In the $N_c = 1$ case it is not difficult to show that the moduli space of the theory is in fact described by a conifold singularity, with the following map between geometrical data and mesonic vevs

$$z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1. \tag{3.3}$$

The conifold is a 6d Calabi-Yau cone over the $T^{1,1}$ Sasaki-Einstein manifold. Its Ricci flat metric is usually written as

$$ds_C^2 = dr^2 + r^2 ds_{T^{1,1}}^2, \tag{3.4}$$

$$ds_{T^{1,1}}^2 = \frac{1}{6} \sum_{i=1}^2 \left[d\theta_i^2 + \sin^2 \theta_i d\varphi_i^2 \right] + \frac{1}{9} \left[d\psi + \sum_{i=1}^2 \cos \theta_i d\varphi_i \right]^2,$$

where the range of the angles is $\psi \in [0, 4\pi)$, $\varphi_i \in [0, 2\pi)$, $\theta_i \in [0, \pi]$. In terms of these coordinates we can write

$$z_1 = r^{\frac{3}{2}} e^{\frac{i}{2}(\psi - \varphi_1 - \varphi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \quad z_2 = r^{\frac{3}{2}} e^{\frac{i}{2}(\psi + \varphi_1 + \varphi_2)} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2},$$

$$z_3 = r^{\frac{3}{2}} e^{\frac{i}{2}(\psi + \varphi_1 - \varphi_2)} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \quad z_4 = r^{\frac{3}{2}} e^{\frac{i}{2}(\psi - \varphi_1 + \varphi_2)} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}. \tag{3.5}$$

For our purposes of generalizing the background to an unquenched setup, it is more convenient to work with a redefined radial coordinate

$$r = r_0 e^\rho, \tag{3.6}$$

where ρ is a dimensionless quantity and r_0 is a constant which, in order to simplify notation, we set to one from now on. The supergravity background sourced by the N_c D3 branes has a metric of the known warped form

$$ds^2 = h^{-1/2}(\rho) [dx_\mu dx^\mu] + h^{1/2}(\rho) [e^{2\rho} d\rho^2 + e^{2\rho} ds_{T^{1,1}}^2], \tag{3.7}$$

constant dilaton ϕ , and N_c units of F_5 RR flux through $T^{1,1}$. In the decoupling limit, where the geometry is dual to the Klebanov-Witten fixed point, the relevant metric is $AdS_5 \times T^{1,1}$.

The D7-branes used to flavor the KW theory, are taken to be extended along the Minkowski 4d directions and along a non compact four-dimensional submanifold in the transverse space. This makes relatively easy the task of solving for the embedding equations. In fact the warp factor drops out in the D7 DBI action and the spacetime effectively seen by the brane is a direct product of Minkowski spacetime and the conifold. This allows to write the embedding equations in a relatively simple way in terms of the conifold coordinates z_i . A detailed study of the holomorphic D7 embeddings on the conifold can be found in [22–25].

Two classes of embeddings, having $z_1 = \mu$ and $z_1 - z_2 = \mu$ as representative elements, were mainly considered in the literature. On the field theory side the two classes correspond to the addition, to the KW superpotential, of extra cubic and quartic terms, respectively, as well as of mass terms for the fundamental flavors when $\mu \neq 0$. In the following we will focus on the first class only. In appendix A we will briefly review the field theory interpretation of both kind of embeddings, starting from an orbifold construction.

3.1 Massless embeddings

The simplest embedding we want to consider is $z_1 = 0$. We call this embedding massless since it is extended down to $\rho = -\infty$. From (3.5) above, it obviously implies $\theta_1 = 0$ or $\theta_2 = 0$. Thus, there are two branches. In each branch, the D7 brane fully wraps one of the two two-spheres of $T^{1,1}$ and chooses a point in the other one. Notice that usually choosing a point in a S^2 requires two equations (fixing θ, φ), but of course if one is at a pole ($\sin \theta = 0$), it is not necessary to specify φ . Thus we can describe a massless embedding of this kind by choosing two points, one for each sphere, fixing the position of each branch.

From the field theory point of view the two branches correspond to two classes of flavor multiplets, which we denote by \tilde{q}_1, q_1 and \tilde{q}_2, q_2 . The addition to the Klebanov-Witten theory of N_f D7-branes with $z_1 = 0$ embedding, modifies the field theory superpotential as [22]

$$W_{z_1=0} = W_{\text{KW}} + h_1 \tilde{q}_1 A_1 q_2 + h_2 \tilde{q}_2 B_1 q_1, \tag{3.8}$$

where, here and in the following, sums over the N_f flavor indices are implied. The classical flavor symmetry preserved by $W_{z_1=0}$ is $U(N_f) \times U(N_f)$. Because of the $SU(2) \times SU(2)$ symmetry, one should be able to pick a (two-branch) embedding by picking any point on each of the spheres. This means that we should have a four-parameter family of massless embeddings. This family is described by the more general embedding equation [22]

$$\sum_{i=1}^4 \alpha_i z_i = 0, \tag{3.9}$$

with the complex constants α_i spanning a conifold (up to overall complex rescalings)

$$\alpha_1 \alpha_2 - \alpha_3 \alpha_4 = 0. \tag{3.10}$$

Notice that embeddings like $z_1 - z_2 = 0$ are not in this family.² In (3.9), (3.10), we can rescale the coefficients to fix³ $\alpha_1 = 1$. Then (3.10) fixes α_2 . Thus, this class of embeddings is parameterized by two complex numbers α_3, α_4 , as we expect in order to be able to choose a point in each of the spheres for each branch. Indeed, the embedding equation

$$z_1 + \alpha_3 \alpha_4 z_2 + \alpha_3 z_3 + \alpha_4 z_4 = 0, \tag{3.11}$$

nicely factorizes into two branches

$$\left(\sin \frac{\theta_1}{2} + \alpha_3 e^{i\varphi_1} \cos \frac{\theta_1}{2} \right) \left(\sin \frac{\theta_2}{2} + \alpha_4 e^{i\varphi_2} \cos \frac{\theta_2}{2} \right) = 0, \tag{3.12}$$

so, as expected, α_3, α_4 determine at which point each branch is sitting in each sphere:

$$\begin{aligned} \theta_1 &= \theta_1^\infty \equiv 2 \arctan |\alpha_3|, & \varphi_1 &= \varphi_1^\infty \equiv \pi - \arg[\alpha_3], \\ \theta_2 &= \theta_2^\infty \equiv 2 \arctan |\alpha_4|, & \varphi_2 &= \varphi_2^\infty \equiv \pi - \arg[\alpha_4]. \end{aligned} \tag{3.13}$$

The constants $\theta_i^\infty, \varphi_i^\infty$ denote the position of each branch as $\rho \rightarrow \infty$ (a notation that will be useful in the next section). We can rewrite (3.12) as

$$\Gamma_1 \Gamma_2 = 0, \tag{3.14}$$

where we have defined

$$\Gamma_i \equiv \cos \frac{\theta_i^\infty}{2} \sin \frac{\theta_i}{2} - e^{i(\varphi_i - \varphi_i^\infty)} \sin \frac{\theta_i^\infty}{2} \cos \frac{\theta_i}{2}, \quad (i = 1, 2). \tag{3.15}$$

For later reference, it is worth remarking that

$$0 \leq |\Gamma_i| \leq 1. \tag{3.16}$$

3.2 Massive embeddings

We now want to give a mass to the flavors. This deforms the embeddings in such a way that the two branches of the massless embeddings merge (the $U(N_f) \times U(N_f)$ flavor symmetry is explicitly broken down to $U(N_f)$). Let us again start with the simplest embedding $z_1 = e^{\frac{3}{2}\rho_q} e^{i\beta}$, where ρ_q, β are real numbers. This is referred to as the unit winding ($n_1 = n_2 = 1$) embedding in the notations of [24] and it is explicitly given by

$$\psi = \varphi_1 + \varphi_2 + 2\beta, \quad e^{3\rho} = \frac{e^{3\rho_q}}{\left(\sin \frac{\theta_1}{2}\right)^2 \left(\sin \frac{\theta_2}{2}\right)^2}. \tag{3.17}$$

Clearly, ρ_q is the minimal value of ρ reached by the brane. This is a connected embedding which for large ρ goes to the two branches of the corresponding massless embedding.

²A $SU(2) \times SU(2)$ rotation, in fact, maps $z_1 - z_2 = 0$ into a generalized embedding equation as in (3.9), with the parameters spanning a unit 3-sphere: $\alpha_1 = -\bar{\alpha}_2, \alpha_3 = \bar{\alpha}_4, |\alpha_1|^2 + |\alpha_3|^2 = 1$.

³Obviously this does not work for the $\alpha_1 = 0$ case, which implies that one of the branches is at the south pole of one sphere. This is a zero measure subset of the embeddings considered.

We now want to consider the same generalized class for the massive embeddings as we used in section 3.1. The defining equation is [22]

$$z_1 + \alpha_3 \alpha_4 z_2 + \alpha_3 z_3 + \alpha_4 z_4 = \text{const} . \tag{3.18}$$

In terms of the angles, we can write it as

$$e^{\frac{3}{2}\rho} e^{\frac{i}{2}(\psi - \varphi_1 - \varphi_2)} \Gamma_1 \Gamma_2 = e^{\frac{3}{2}\rho_q} e^{i\beta} , \tag{3.19}$$

with the same Γ 's defined in (3.15). Something obvious that is worth noting is that the embedding equation is not independent on ψ any more, as expected because a mass term breaks the $U(1)_R$ symmetry. Shifting ψ by a constant shifts the phase β . Consistently, there is a $\frac{1}{2}$ in front of ψ such that a 4π shift which takes us to the same point, shifts the mass phase by 2π , thus leaving it invariant.

Thus, the embedding of a brane within this family depends on 6 real parameters: θ_1^∞ , φ_1^∞ , θ_2^∞ , φ_2^∞ , β , ρ_q . In the field theory, the first four should be identified with the couplings in the superpotential while the last two should be related to the phase and the modulus of the mass term.

The generalized embeddings we have found above, can be mapped to a field theory superpotential of the form

$$W = W_{\text{KW}} + h_1 \tilde{q}_1 [A_1 + \alpha_4 A_2] q_2 + h_2 \tilde{q}_2 [B_1 + \alpha_3 B_2] q_1 + m_1 \tilde{q}_1 q_1 + m_2 \tilde{q}_2 q_2 , \tag{3.20}$$

where we put generic m_1, m_2 mass terms (then we will want $m_1 = m_2$). We can formally rewrite W in a compact form as $W = W_{\text{KW}} + \tilde{q} M q$ where M is the effective mass matrix for the flavors. Following the arguments in [22], the equation $\det M = 0$ should correspond to the D7 embedding equation. Indeed, it reads

$$h_1 h_2 [A_1 + \alpha_4 A_2] [B_1 + \alpha_3 B_2] = m_1 m_2 , \tag{3.21}$$

which is equivalent (using, for the $N_c = 1$ case, the maps (3.3)) to the equation

$$z_1 + \alpha_3 \alpha_4 z_2 + \alpha_3 z_3 + \alpha_4 z_4 = \frac{m_1 m_2}{h_1 h_2} . \tag{3.22}$$

This is exactly of the form we proposed above, for the massless $m_1 = m_2 = 0$ (3.11) and the massive (3.18) case.

4. Smearing the embeddings

We want now to consider $N_f \gg 1$ branes suitably distributed within the family of embeddings (3.19). In particular, we want to restore effectively the $SU(2) \times SU(2) \times U(1)$ symmetry of $T^{1,1}$ which is broken by flavor branes on top of each other.

Accordingly, we want to place branes at different values of $\theta_1^\infty, \varphi_1^\infty, \theta_2^\infty, \varphi_2^\infty, \beta$, while, in principle, we are interested in keeping ρ_q fixed (clearly, it is also possible to implement a distribution of ρ_q 's). The symmetry preserving density of branes is

$$\rho_{\theta_1^\infty \varphi_1^\infty \theta_2^\infty \varphi_2^\infty \beta} \equiv \frac{dn}{d\theta_1^\infty d\varphi_1^\infty d\theta_2^\infty d\varphi_2^\infty d\beta} = \frac{N_f}{2\pi (4\pi)^2} \sin \theta_1^\infty \sin \theta_2^\infty , \tag{4.1}$$

which is normalized as

$$\int_0^\pi d\theta_1^\infty \int_0^{2\pi} d\varphi_1^\infty \int_0^\pi d\theta_2^\infty \int_0^{2\pi} d\varphi_2^\infty \int_0^{2\pi} d\beta \rho_{\theta_1^\infty \varphi_1^\infty \theta_2^\infty \varphi_2^\infty \beta} = N_f . \quad (4.2)$$

The purpose of this section is to compute how the charge density produced by this D7-brane distribution modifies the Bianchi identity for F_1 . This coupling comes from the WZ term, which can be written as (cfr. eq (3.1) of [9])

$$S_{\text{WZ}} = T_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 \rightarrow T_7 \int_{\mathcal{M}_{10}} \Omega \wedge C_8 , \quad (4.3)$$

yielding:

$$dF_1 = -T_7(2\kappa_{(10)}^2)\Omega = -g_s\Omega . \quad (4.4)$$

4.1 General procedure

Let us start by considering a generic distribution of D7-branes. The Ω defined above is built from the orthogonal planes to the submanifolds where the branes are sitting. Each brane is described by two equations

$$f_1(p_i; x^\mu) = 0 , \quad f_2(p_i; x^\mu) = 0 , \quad (4.5)$$

where the p_i are some parameters and x^μ are the spacetime coordinates ($\mu = 0, \dots, 9$). Locally, the orthogonal plane to a single embedding is described by a two-form:

$$\delta(f_1)\delta(f_2)df_1 \wedge df_2 . \quad (4.6)$$

We now need to sum over all the branes. This becomes just an integral over the parameters which define the embeddings, suitably weighted with the density:

$$\Omega = \int \rho_{p_i}(p_i) (\delta(f_1)\delta(f_2)df_1 \wedge df_2) dp_i . \quad (4.7)$$

4.1.1 A simple example

To see how the calculation is performed, let us work out the expression (4.7) for a case in which we already know the answer: the smeared massless embedding in the KW case, as discussed in [9]. There, since each embedding is disconnected into two branches, we have to compute separately each contribution.

Consider the collection of branches described by

$$f_1(\theta_1^\infty, \varphi_1^\infty; x^\mu) = \theta_1 - \theta_1^\infty , \quad f_2(\theta_1^\infty, \varphi_1^\infty; x^\mu) = \varphi_1 - \varphi_1^\infty . \quad (4.8)$$

The distribution density is

$$\rho_{\theta_1^\infty \varphi_1^\infty} = \frac{N_f}{4\pi} \sin \theta_1^\infty . \quad (4.9)$$

We plug this in (4.7) to get

$$\Omega = \left(\int \frac{N_f}{4\pi} \sin \theta_1^\infty \delta(\theta_1 - \theta_1^\infty) \delta(\varphi_1 - \varphi_1^\infty) d\theta_1^\infty d\varphi_1^\infty \right) d\theta_1 \wedge d\varphi_1 = \frac{N_f}{4\pi} \sin \theta_1 d\theta_1 \wedge d\varphi_1 . \quad (4.10)$$

From the other branch, we get a similar contribution with ($1 \rightarrow 2$), so $\Omega = \frac{N_f}{4\pi} (\sin \theta_1 d\theta_1 \wedge d\varphi_1 + \sin \theta_2 d\theta_2 \wedge d\varphi_2)$. This, together with (4.4) above, agrees with equation (2.13) of [9].

4.2 The Bianchi identity for the smeared massive case

We now want to compute the expression (4.7) where f_1, f_2 can be read from (3.19) and the density is written in (4.1). Let us split the modulus and the phase in (3.19). From equating the phases, we get

$$f_1 = \psi - \varphi_1 - \varphi_2 + 2 \arg(\Gamma_1) + 2 \arg(\Gamma_2) - 2\beta - 4\pi n, \quad (4.11)$$

where n is any integer number.

From the modulus (squared) we find

$$f_2 = e^{3\rho} |\Gamma_1|^2 |\Gamma_2|^2 - e^{3\rho_q}, \quad (4.12)$$

where

$$|\Gamma_i|^2 = \frac{1}{2} [1 - \cos \theta_i^\infty \cos \theta_i - \sin \theta_i \sin \theta_i^\infty \cos(\varphi_i - \varphi_i^\infty)]. \quad (4.13)$$

We now have to insert these expressions, together with (4.1), into (4.7). Let us start by making the integral in β . Since the only dependence from β in the integrand comes from $\delta(f_1)$, we have $\int \delta(f_1) d\beta = \frac{1}{2}$ (notice that f_1 changes by 4π when β goes from 0 to 2π , so, for any values of $\psi, \theta_i, \varphi_i, \theta_i^\infty, \varphi_i^\infty$, the argument of the delta function is zero exactly one time in the β range of integration). Thus:

$$\Omega = \frac{N_f}{(4\pi)^3} \int (\sin \theta_1^\infty \sin \theta_2^\infty \delta(f_2) df_1 \wedge df_2) d\theta_1^\infty d\varphi_1^\infty d\theta_2^\infty d\varphi_2^\infty. \quad (4.14)$$

Notice that, because of (3.16), f_2 can only vanish if $\rho > \rho_q$. Therefore, $\Omega(\rho < \rho_q) = 0$, as expected, since the flavor branes are extended in $\rho > \rho_q$ only.

The integral (4.14) seems extremely difficult to compute. However, the result must be fairly simple, just of the form displayed in (3.99) of [9] for a certain⁴ $N_f(\rho)$ (a dot will denote derivative with respect to ρ throughout the paper):

$$\Omega = \frac{N_f(\rho)}{4\pi} (\sin \theta_1 d\theta_1 \wedge d\varphi_1 + \sin \theta_2 d\theta_2 \wedge d\varphi_2) - \frac{\dot{N}_f(\rho)}{4\pi} d\rho \wedge (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2). \quad (4.15)$$

This is the only possibility for an exact two-form preserving $SU(2) \times SU(2) \times U(1)_\psi \times \mathbb{Z}_2$ (the \mathbb{Z}_2 interchanging the 1 and 2 S^2 's). This observation will allow us to obtain an exact simple expression for Ω since this task is just reduced to computing a single function $N_f(\rho)$.

Let us expand $\Omega = \frac{1}{2} \Omega_{MN} dx^M \wedge dx^N$, where the M, N are the coordinates $\rho, \psi, \theta_1, \theta_2, \varphi_1, \varphi_2$. We can easily isolate the different terms of this expression and compute them in turn. The simplest component is

$$\Omega_{\rho\psi} = -\frac{N_f}{(4\pi)^3} \int (\sin \theta_1^\infty \sin \theta_2^\infty \delta(f_2) 3e^{3\rho} |\Gamma_1|^2 |\Gamma_2|^2) d\theta_1^\infty d\varphi_1^\infty d\theta_2^\infty d\varphi_2^\infty. \quad (4.16)$$

Still, this is not so easy to compute due to the delta function. But, since $\Omega_{\rho\psi}$ cannot depend on θ_i, φ_i (see (4.15)), we can fix $\theta_1 = \theta_2 = 0$ such that, from (4.13), we have

⁴Notice that N_f will denote the fixed number of flavor branes while $N_f(\rho)$ is a non-trivial function, representing the effective number of massless flavors at a given energy scale. These two quantities should not be confused, they are only equal in the massless case, $\rho_q = -\infty$.

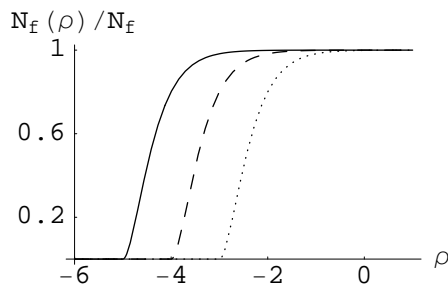


Figure 1: The function $N_f(\rho)$ for three different values of the mass parameter (resp. from left to right) $\rho_q = -5, -4, -3$.

$|\Gamma_i|^2 = \frac{1}{2}(1 - \cos \theta_i^\infty)$. Changing variables to $a_i = \frac{1}{2}(1 - \cos \theta_i^\infty)$ (notice that $0 \leq a_i \leq 1$), the integral above now takes a simple form:

$$\Omega_{\rho\psi} = -\frac{N_f}{4\pi} \int \delta(e^{3\rho} a_1 a_2 - e^{3\rho_q}) 3e^{3\rho} a_1 a_2 da_1 da_2 = -\frac{3N_f}{4\pi} e^{3\rho_q - 3\rho} \int_{e^{3\rho_q - 3\rho}}^1 \frac{da_2}{a_2}. \quad (4.17)$$

Comparing the result to (4.15) we get the sought result:

$$\begin{aligned} \dot{N}_f(\rho) &= 3N_f e^{3\rho_q - 3\rho} (3\rho - 3\rho_q), \\ N_f(\rho) &= N_f [1 - e^{3\rho_q - 3\rho} (1 + 3\rho - 3\rho_q)], \quad (\rho > \rho_q) \end{aligned} \quad (4.18)$$

whereas $N_f(\rho) = 0$ at $\rho < \rho_q$. Consistently, $N_f(\infty) = N_f$ and $N_f(\rho_q) = 0$, so $N_f(\rho)$ is a continuous function which asymptotes to the one of the massless case when $\rho \gg \rho_q$. Notice also that $N_f(\rho), \dot{N}_f(\rho) \geq 0$ as required on intuitive grounds and also by the supergravity equations of motion [9]. As it is shown in figure 1, $N_f(\rho)$ has the shape of a smoothed-out Heaviside step function.

This completes the computation of the Ω since the symmetry constrains the rest of the components to be given by (4.15). We have checked the result (4.18) by calculating explicitly, by numerical integration, the other components.

The effective running of the number of flavors has a natural field theory interpretation. At energy scales larger than the mass of the matter multiplets, the theory looks like the massless one. At energies lower than the mass scale, the massive flavors can be integrated out and the theory resembles the unflavored ($N_f = 0$) one. At scales around $\rho \sim \rho_q$, the non-trivial profile of the function should take into account threshold effects.

5. The backreacted solution with massive flavors

Knowing the precise expression for $N_f(\rho)$, we can find the backreacted solution dual to the massive flavored KW theory using the expressions of [9]. The ansatz for the Einstein frame metric

$$ds^2 = h^{-\frac{1}{2}} dx_{1,3}^2 + h^{\frac{1}{2}} \left(e^{2f} d\rho^2 + \frac{e^{2g}}{6} \sum_{i=1,2} (d\theta_i + \sin^2 \theta_i d\varphi_i^2) + \frac{e^{2f}}{9} \left(d\psi + \sum_{i=1,2} \cos \theta_i d\varphi_i \right)^2 \right), \quad (5.1)$$

is a deformation, driven by the functions $f(\rho), g(\rho)$, of the warped conifold considered up to now (for which $f(\rho) = g(\rho) = \rho$). The radial coordinate ρ varies in $(-\infty, \rho_L]$ with the lower (resp. upper) extremum being mapped, via the holographic radius/energy relation, to the extreme IR (resp. UV Landau pole) of the dual field theory. The ansatz for the RR-forms and the dilaton reads (we set $g_s = 1$ in the following)

$$\begin{aligned}
 \phi &= \phi(\rho) , \\
 F_5 &= 27\pi N_c \alpha'^2 e^{-4g-f} h(\rho)^{-5/4} \left(e^{x^0 x^1 x^2 x^3 \rho} - e^{\theta_1 \varphi_1 \theta_2 \varphi_2 \psi} \right) , \\
 F_1 &= \frac{N_f(\rho)}{4\pi} (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2) , \\
 dF_1 &= -\frac{N_f(\rho)}{4\pi} (\sin \theta_1 d\theta_1 \wedge d\varphi_1 + \sin \theta_2 d\theta_2 \wedge d\varphi_2) + \\
 &\quad + \frac{\dot{N}_f(\rho)}{4\pi} d\rho \wedge (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2) . \tag{5.2}
 \end{aligned}$$

The vielbein is given by

$$\begin{aligned}
 e^{x^i} &= h^{-1/4} dx^i , & e^\rho &= h^{1/4} e^f d\rho , \\
 e^{\theta_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g d\theta_i , & e^{\varphi_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g \sin \theta_i d\varphi_i , \\
 e^\psi &= \frac{1}{3} h^{1/4} e^f (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2) . \tag{5.3}
 \end{aligned}$$

It is crucial to notice that the projections that define the Killing spinors have the same form for the deformed or undeformed ansatz [9]. Then, it is not difficult to check that the κ -symmetry analysis of [24] is easily generalized to the backreacted solution, and (3.17) is obtained without modification (when one uses the ρ -coordinate). This means that the computation of section 4 and in particular the result (4.18) is directly applicable to the backreacted case.

The other crucial observation is that passing from N_f to $N_f(\rho)$ does not alter the form of the first order equations for the functions appearing in the ansatz. This is due to the fact that the supersymmetric fermionic variations only contain the forms F and not the dF terms. Thus, although the Bianchi identity for F_1 is modified in this massive setup w.r.t. the massless one, we can just use the first order equations found in [9] with $N_f \rightarrow N_f(\rho)$. We thus have:

$$\begin{aligned}
 \dot{g} &= e^{2f-2g} , \\
 \dot{f} &= 3 - 2e^{2f-2g} - \frac{3N_f(\rho)}{8\pi} e^\phi , \\
 \dot{\phi} &= \frac{3N_f(\rho)}{4\pi} e^\phi , \\
 \dot{h} &= -27\pi N_c \alpha'^2 e^{-4g} . \tag{5.4}
 \end{aligned}$$

These equations have to be solved separately for $\rho < \rho_q$ and $\rho > \rho_q$. Then, one has to demand continuity at $\rho = \rho_q$. This will be the content of the upcoming subsections.

The constituent mass m_q for the dynamical quarks can be related to ρ_q by defining it, as usual, as the energy of a straight string stretched along the radial direction from ρ_q to the bottom of the space, that is at $\rho = -\infty$ in our case:

$$m_q = \frac{1}{2\pi\alpha'} \int_{-\infty}^{\rho_q} e^{\frac{\phi}{2}+f} d\rho . \tag{5.5}$$

5.1 Region 1: $\rho < \rho_q$

In this region, one just has the well known unflavored system, after inserting $N_f(\rho) = 0$ in (5.4). The general solution for ϕ, g, f can be given in a simple form:

$$\begin{aligned} e^\phi &= e^{\phi_{\text{IR}}} , \\ e^g &= \tilde{c}_3 (e^{6\rho} + \tilde{c}_1)^{\frac{1}{6}} , \\ e^f &= \tilde{c}_3 e^{3\rho} (e^{6\rho} + \tilde{c}_1)^{-\frac{1}{3}} , \\ h(\rho) &= 27\pi N_c \alpha'^2 \left(\tilde{c}_2 + \int_{\rho}^{\rho_q} e^{-4g(\rho_*)} d\rho_* \right) . \end{aligned} \tag{5.6}$$

For the integration constants, we have used a notation similar to [9].

5.2 Region 2: $\rho > \rho_q$

The relevant system of equations for this region comes from inserting $N_f(\rho)$ as given in (4.18) in (5.4). Remarkably, it turns out that the functions f, g, ϕ can be explicitly integrated in this region too. Just as in the massless case, the dilaton is running and blows up at a certain ρ_L . Fixing one of the constants of integration, which just amounts to shifting ρ, ρ_q , we require, as in [9], that the dilaton divergence (i.e. the field theory UV Landau pole) is located at $\rho_L = 0$. The solution is:

$$\begin{aligned} e^\phi &= -\frac{4\pi}{N_f(3\rho - e^{3\rho_q - 3\rho}(3\rho_q - 3\rho - 2) + e^{3\rho_q}(3\rho_q - 2))} , \\ e^f &= c_3 \frac{(-6e^{6\rho}\rho + 2e^{3\rho+3\rho_q}(3\rho_q - 3\rho - 2) - 2e^{6\rho+3\rho_q}(3\rho_q - 2))^{\frac{1}{2}}}{(e^{6\rho}(1 - 6\rho) + 4e^{3\rho+3\rho_q}(3\rho_q - 3\rho - 1) - 2e^{6\rho+3\rho_q}(3\rho_q - 2) + c_1)^{\frac{1}{3}}} , \\ e^g &= c_3 (e^{6\rho}(1 - 6\rho) + 4e^{3\rho+3\rho_q}(3\rho_q - 3\rho - 1) - 2e^{6\rho+3\rho_q}(3\rho_q - 2) + c_1)^{\frac{1}{6}} , \\ h(\rho) &= 27\pi N_c \alpha'^2 \left(c_2 + \int_{\rho}^0 e^{-4g(\rho_*)} d\rho_* \right) . \end{aligned} \tag{5.7}$$

5.3 Patching the solutions at $\rho = \rho_q$

Out of the seven integration constants appearing in (5.6), (5.7), four are fixed by demanding

continuity of the functions at $\rho = \rho_q$, namely:

$$\begin{aligned}
 e^{\phi_{\text{IR}}} &= \frac{4\pi}{N_f (e^{3\rho_q}(2 - 3\rho_q) - (2 + 3\rho_q))} , \\
 c_1 &= -e^{6\rho_q} + \tilde{c}_1 (2 e^{3\rho_q}(2 - 3\rho_q) - 2(2 + 3\rho_q)) , \\
 \tilde{c}_3 &= c_3 (2 e^{3\rho_q}(2 - 3\rho_q) - 2(2 + 3\rho_q))^{\frac{1}{6}} , \\
 \tilde{c}_2 &= c_2 + \int_{\rho_q}^0 e^{-4g(\rho_*)} d\rho_* .
 \end{aligned}
 \tag{5.8}$$

Notice that the infrared dilaton blows up, signaling the breaking of the validity of the supergravity approximation, for $\rho_q \rightarrow 0$, i.e. for dynamical masses close to the Landau pole.

5.4 A prescription for the integration constants

Apart from depending on the parameters N_c, N_f, ρ_q , which have a clear field theory interpretation (ρ_q being related to the modulus of the quark masses), the solution described above also contains three independent integration constants \tilde{c}_1, c_2, c_3 . In this subsection we will discuss their physical meaning and give a prescription to fix them when comparing different solutions, as we will do in section 7.

Let us start with \tilde{c}_1 , which severely affects the IR behaviors. If $\tilde{c}_1 = 0$ one recovers in the IR limit the standard regular solution for D3-branes on the conifold. If $\tilde{c}_1 > 0$ the solution is of the form discussed in [26]: in this case the transverse space to the D3-branes is a deformation of the (\mathbb{Z}_2 orbifold of the) conifold where a 4-cycle has blown up. The D3's are actually smeared along that cycle and their backreaction gives rise to a singular 10d background. Here, we will not be interested in such solutions and thus, by demanding IR regularity, we impose

$$\tilde{c}_1 = 0 ,
 \tag{5.9}$$

which, from (5.8), sets

$$c_1 = -e^{6\rho_q} .
 \tag{5.10}$$

The constant c_2 affects mainly the UV and corresponds to turning on a source term for an irrelevant operator. We will require our solutions to share a common UV behavior by setting

$$c_2 = 0 ,
 \tag{5.11}$$

such that h precisely vanishes at the Landau pole. We then have $\phi|_{\rho=0} = \infty$, $h|_{\rho=0} = e^f|_{\rho=0} = 0$.

Finally, c_3 , which simultaneously rescales $e^{2g}, e^{2f}, h^{-\frac{1}{2}}$, can be reabsorbed by a rescaling of the Minkowski coordinates. We will adopt an *ad hoc* prescription and, as before, require that all solutions behave similarly in the UV by demanding $e^g|_{\rho=0} = 1$. This fixes:

$$c_3 = (1 - e^{6\rho_q} + 6\rho_q e^{3\rho_q})^{-\frac{1}{6}} .
 \tag{5.12}$$

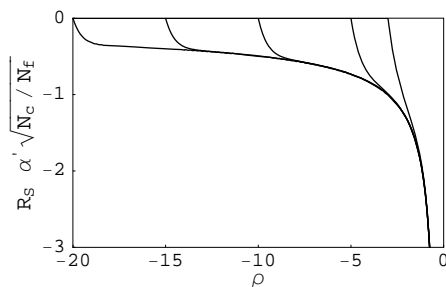


Figure 2: The curvature scalar as a function of ρ for, from left to right, $\rho_q = -20, -15, -10, -5, -3$.

Inserting this value in (5.8) we obtain:

$$\tilde{c}_3 = \left(\frac{2 e^{3\rho_q} (2 - 3\rho_q) - 2(2 + 3\rho_q)}{1 - e^{6\rho_q} + 6\rho_q e^{3\rho_q}} \right)^{\frac{1}{6}}. \quad (5.13)$$

Let us close this section by inserting these prescriptions in the definition of the constituent quark mass (5.5):

$$m_q = \frac{1}{2\pi\alpha'} \sqrt{\frac{8\pi}{N_f}} \frac{e^{\rho_q}}{(1 - e^{6\rho_q} + 6\rho_q e^{3\rho_q})^{\frac{1}{6}} (2 e^{3\rho_q} (2 - 3\rho_q) - 2(2 + 3\rho_q))^{\frac{1}{3}}}. \quad (5.14)$$

Notice that $\lim_{\rho_q \rightarrow 0} m_q = \infty$. Of course we cannot trust the supergravity formulas in this limit since the dilaton is blowing up. The best we can say is that these results suggest that, within the choice of integration constants adopted above, we can explore a wide range of dynamical mass parameters, from $m_q \rightarrow 0$ to some higher $m_q < \Lambda_{UV}$.

5.5 The Ricci curvature and regime of validity

The validity of the supergravity approximation that we are using, requires the curvature invariants of the string frame version of (5.1) to be small in α' units. Let us focus, in particular, on the Ricci scalar. Using the BPS equations (5.4), one obtains

$$R_S = -2 \frac{3N_f(\rho)}{4\pi} h^{-\frac{1}{2}} e^{-2g + \frac{\phi}{2}} \left(7 + 4 \frac{3N_f(\rho)}{4\pi} e^{2g - 2f + \phi} + \frac{7}{4} e^{2g - 2f} \frac{\dot{N}_f(\rho)}{N_f(\rho)} \right). \quad (5.15)$$

Obviously, it vanishes in region 1 ($\rho < \rho_q$) where the near-horizon ($\rho \rightarrow -\infty$) metric is $AdS_5 \times T^{1,1}$ and the dilaton is constant. Validity of supergravity in region 1 requires $\lambda_{IR} = N_c e^{\phi_{IR}}$ to be large and $e^{\phi_{IR}}$ to be small, as usual. From (5.8) we see that these conditions are satisfied, at fixed ρ_q , if $N_c \gg N_f \gg 1$.

Let us now concentrate on region 2. Since $e^\phi \sim N_f^{-1}$ and $h \sim N_c \alpha'^2$, it is easy to see that one can write R_S as $\sqrt{N_f/N_c} \alpha'^{-1}$ times a factor which only depends on ρ, ρ_q (once the integration constants c_1, c_2 have been fixed, notice that R_S does not depend on c_3). Figure 2 depicts some examples, where the prescription of section 5.4 for c_1, c_2 has been used.

In region 2, in order for the supergravity approximation to be valid in a large range of ρ , we need $N_f \ll N_c$ so that the Ricci scalar stays parametrically small except, eventually, very near the Landau pole.⁵ Notice also that the only singularity of the solution is located at the Landau pole (a pathological behavior expected on physical grounds). As stressed in the introduction, the smearing of massive flavors allows to smooth out IR singularities.

Apart from checking the validity of the supergravity approximation which requires $N_c \gg N_f \gg 1$, we should also check the validity of the smeared approximation and the DBI action, along the lines of section 2. The branes we are smearing have codimension $d = 2$ and, as we have seen, the typical length of the transverse space is governed by $R \sim (N_c/N_f)^{\frac{1}{4}}$. Within an area of α' size there are $N_f/R^2 \approx N_f^{\frac{3}{2}} N_c^{-\frac{1}{2}}$ flavor branes, a number which should be larger than 1 for the smearing to be a good approximation down to α' scales. On the other hand, for the DBI to be valid, $e^\phi N_f^{\frac{3}{2}} N_c^{-\frac{1}{2}} \ll 1$. Since $e^\phi \sim N_f^{-1}$, it is clear that this condition does not impose any further restriction. We thus get the following range of parameters:

$$1 \ll N_c^{\frac{1}{3}} \ll N_f \ll N_c. \tag{5.16}$$

Similar restrictions on the parameters should be considered also in the massless case $\rho_q = -\infty$. It should be interesting to investigate if there are regions of ρ for which our solution can be valid for more general values of N_f/N_c and to check how the curvature - and, so, the validity range of the supergravity approximation - is affected by the choice of the integration constants.

5.6 Further comments on the dual field theory

As we have remarked above, the string background we have constructed is dual to a flavored Klebanov-Witten theory which has gauge group $SU(N_c) \times SU(N_c)$, a Landau pole in the UV and is conformal in the IR. This is reflected by the behavior of the dilaton e^ϕ , which, provided the standard orbifold dictionary can be applied to the present setup, is mapped to the field theory gauge couplings by

$$\frac{4\pi}{g_1^2} + \frac{4\pi}{g_2^2} = e^{-\phi}. \tag{5.17}$$

Without loss of generality here we consider the case $g_1^2 = g_2^2 \equiv g_{\text{FT}}^2 = 8\pi e^\phi$. The dilaton is running and going to infinity for $\rho \rightarrow 0$, while it is constant in the region $\rho < \rho_q$. The beta function for g_{FT} can be inferred, holographically, from the dilaton equation

$$\frac{d}{d\rho} e^{-\phi} = -\frac{3N_f(\rho)}{4\pi}, \tag{5.18}$$

once the precise radius/energy relation is given. In the massless case, the relation $\rho = \log(\mu/\Lambda_{\text{UV}})$ was used to get the field theory beta function [9]. In our case this relation

⁵This restriction is typical in D3D7 systems [1]. Notice that, despite the limit $N_f \ll N_c$, we are not in the quenched approximation, since the backreaction of the flavor branes is taken into account.

should still be valid with a good approximation when the flavors are very light. Notice that this relation suggests that

$$\rho_q = \log \frac{m_q}{\Lambda_{UV}} , \tag{5.19}$$

which, looking at eq. (5.14), is just the leading term in the small mass limit where $\rho_q \rightarrow -\infty$. Within these approximations, we can write the field theory running coupling as

$$\frac{8\pi}{g_{\text{FT}}^2} \approx -\frac{N_f}{4\pi} \left[3 \log \frac{\mu}{\Lambda_{UV}} - \left(\frac{m_q}{\mu}\right)^3 \left[3 \log \frac{m_q}{\mu} - 2 \right] + \left(\frac{m_q}{\Lambda_{UV}}\right)^3 \left[3 \log \frac{m_q}{\Lambda_{UV}} - 2 \right] \right] . \tag{5.20}$$

In this expression, the first term reproduces the one-loop result, while the rest is a string prediction.⁶ The power-like factors in the above expression could be interpreted in field theory as fractional instantons, or renormalon corrections; the interesting feature is the presence of just two such terms.

6. The Heaviside approximation

A simple way to model the integrating in/out of the massive flavor degrees of freedom in field theory is to forget about the details of the embedding and model the string dual with a Heaviside step function $N_f(\rho) = N_f \Theta(\rho - \rho_q)$. Of course this approximation is a source of systematic errors, due to the differences between the step function and the true value of $N_f(\rho)$ extracted from the smearing procedure. In the present setup we can measure these errors, comparing (see the following sections) physical observables deduced using both the true and the approximate supergravity solutions. Estimating these errors is of notable importance to have an idea of how well the step function approximation can work in general models like [7], where the correct $N_f(\rho)$ may be hard to compute.

We now present the solution using the Heaviside form for $N_f(\rho)$, using capital symbols for integration constants within this framework. At $\rho < \rho_q$, we still have (5.6), whereas for $\rho > \rho_q$ the solution reads:

$$\begin{aligned} e^\phi &= -\frac{4\pi}{3N_f\rho} , \\ e^g &= C_3 \left((1 - 6\rho)e^{6\rho} + C_1 \right)^{\frac{1}{6}} , \\ e^f &= C_3 \sqrt{-6\rho} e^{3\rho} \left((1 - 6\rho)e^{6\rho} + C_1 \right)^{-\frac{1}{3}} , \\ h &= 27\pi N_c \alpha'^2 \left(C_2 + \int_\rho^0 e^{-4g(\rho^*)} d\rho^* \right) . \end{aligned} \tag{6.1}$$

As in section 5.3, continuity at ρ_q gives some relations among the different constants:

$$e^{\phi_{\text{IR}}} = -\frac{4\pi}{3N_f\rho_q} , \quad C_1 = -6\rho_q \tilde{C}_1 - e^{6\rho_q} , \quad \tilde{C}_3 = C_3 (-6\rho_q)^{\frac{1}{6}} , \tag{6.2}$$

⁶One has to keep in mind that the string theory calculation of $N_f(\rho)$ is reflecting a particular choice of renormalization scheme in the field theory side.

while if we follow again the prescriptions of section 5.4, we get:

$$\tilde{C}_1 = C_2 = 0, \quad C_3 = (1 - e^{6\rho_q})^{-\frac{1}{6}}. \quad (6.3)$$

Under this prescription, the constituent quark mass (5.5) reads:

$$m_q = \frac{1}{2\pi\alpha'} \sqrt{\frac{8\pi}{N_f}} \frac{e^{\rho_q}}{(-6\rho_q)^{\frac{1}{3}}(1 - e^{6\rho_q})^{\frac{1}{6}}}. \quad (6.4)$$

Notice that for $\rho_q \rightarrow -\infty$ the solutions obtained with the actual $N_f(\rho)$ approach those written above: this suggests that the Heaviside approximation is quite good in the small m_q limit.

7. Wilson loops in backreacted D3D7 models

In the previous sections we have constructed a string dual of a flavored version of the Klebanov-Witten theory with a large number N_f of dynamical flavors of mass m_q . Now we want to study how these sea flavors affect the non perturbative dynamics of the gauge theory. We are going to probe the latter with an external quark-antiquark \bar{Q}, Q pair with mass $M_Q \gg m_q$. Due to the presence of a Landau pole in the gauge theory, M_Q cannot be taken to be infinite and it must lie below the pole. The idea is to study how the static quark-antiquark potential depends on the sea quark parameters m_q, N_f . The effects of N_f flavors on the energy/spin relations for mesonic-like bound states in a similar but localized D3D7 system have been studied in [27].

The $\bar{Q}Q$ bound state is dual to an open string with the extrema lying on a probe D7-brane embedded in such a way that it reaches a minimal distance $\rho_Q \gg \rho_q$ from the bottom of the space ($M_Q = \frac{1}{2\pi\alpha'} \int_{-\infty}^{\rho_Q} e^{\frac{\phi}{2}+f} d\rho$). The string, in turn, bends in the bulk and reaches a minimal radial position ρ_0 . The Minkowski separation L between the test quarks, as well as the total energy of the system, depends on ρ_0 . From this we can deduce the $V(L)$ relation, where $V(L)$ is the $\bar{Q}Q$ potential, i.e. the total energy to which the total contribution from the quark mass $2M_Q$ has been subtracted. The open string embedding is chosen as $t = \tau, y = \sigma, \rho = \rho(y)$ where $y \in [-L/2, L/2]$ is one of the spatial Minkowski directions. The string worldsheet action reads

$$S = -\frac{1}{2\pi\alpha'} \int dt dy \sqrt{g_{tt}(\rho)[g_{yy}(\rho) + (\partial_y \rho)^2 g_{\rho\rho}(\rho)]},$$

where $g_{\mu\nu}$ refers here to the string frame metric, which is obtained multiplying (5.1) by $e^{\frac{\phi}{2}}$. Defining

$$F = \sqrt{g_{tt}g_{yy}} = e^{\frac{\phi}{2}} h^{-1/2}, \quad G = \sqrt{g_{tt}g_{\rho\rho}} = e^{\frac{\phi}{2}} e^f, \quad (7.1)$$

we can write [28, 29] the string length and renormalized energy as (the 0 subindex means that the quantity is evaluated at $\rho = \rho_0$)

$$L(\rho_0) = 2 \int_{\rho_0}^{\rho_Q} \frac{GF_0}{F \sqrt{F^2 - F_0^2}} d\rho, \\ V(\rho_0) = \frac{2}{2\pi\alpha'} \left[\int_{\rho_0}^{\rho_Q} \frac{GF}{\sqrt{F^2 - F_0^2}} d\rho - \int_0^{\rho_Q} G d\rho \right].$$

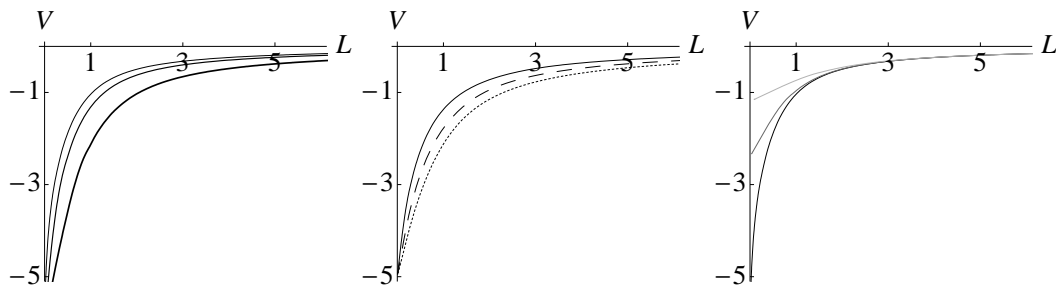


Figure 3: From left to right, plots of $V(L)$ at different values of: $N_f = 1, 0.6, 0.2$, top to bottom (at $m_q = 0.1, M_Q = 3$); $m_q = 0.5, 1, 1.5$, top to bottom (at $N_f = 1, M_Q = 2.5$); $M_Q = 0.6, 1.2, 3$, top to bottom (at $N_f = 1, m_q = 0.1$). Note that $V(0) = -2M_Q$, so the potential is always bounded from below.

We can use the background solutions found in the previous sections to study how the external quark interaction depends on the dynamical massive flavors. In doing so, we vary one of the physical parameters N_f, m_q, M_Q while keeping the other two fixed. In the numerical computations we will present, lengths will be measured in units of $\sqrt{27\pi N_c \alpha'^2}$ whereas energies in units of $(2\pi\alpha')^{-1}$ (the r_0 that was set to 1 in (3.6) should be reinserted to get the correct dimensions).

From figure 3 we can see that the static quark-antiquark potential is always negative and has the same qualitative Coulomb-like behavior as in the unflavored KW conformal case. This behavior is not unexpected: in the far IR (i.e. at large L) the quarks are integrated out and the theory looks like the KW one. Hence at large L we expect the Coulomb-like behavior typical of a conformal theory. Figure 3 also tell us that the absolute value of the potential $|V(L)|$ at fixed L is a decreasing function of N_f and an increasing function of m_q . This is an effect of the screening of the color charges due to the dynamical flavors: the more the theory is “unquenched” (large N_f , small m_q) the more the modulus of the quark-antiquark force is reduced. Finally, the large L behavior of $V(L)$ is not strongly affected by the choice of the cutoff M_Q ; in the small L region, instead, we see that $|V(L)|$ is an increasing function of M_Q . We will see in section 7.4 how these results nicely fit with the expected qualitative behavior of $V(L)$.

For all the plots of this section, we have adopted the prescription for integration constants given in section 5.4. It can also be instructive to study how the behavior of the $\bar{Q}Q$ interaction is affected by the choice of these integration constants. A brief analysis of the dependence on c_2 can be found in appendix B.

7.1 The screening length

The connected $\bar{Q}Q$ configuration we have studied above is expected to be unstable due to the presence of the dynamical flavors. If the $\bar{Q}Q$ string has enough energy, in fact, a dynamical $\bar{q}q$ pair can be popped out from the vacuum with a consequent decay of the heavy $\bar{Q}Q$ state into a pair of heavy-light mesonic-like bound states $\bar{Q}q + \bar{q}Q$. The length at which the decay can happen is the so called “screening length” L_s . In the model at hand, due to the smearing procedure, the lighter heavy-light “mesons” are nearly massless.

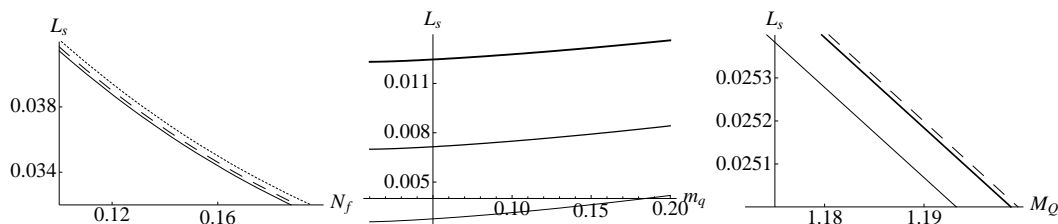


Figure 4: From left to right, the screening length L_s as a function of: N_f for $m_q = 0.01, 0.05, 0.1$, bottom to top ($M_Q = 2$); m_q for $N_f = 1, 1.5, 2$, top to bottom ($M_Q = 2$); M_Q for $N_f = 1, m_q = 0.1$ (thick), $N_f = 1, m_q = 0.101$ (dashed), $N_f = 1.01, m_q = 0.1$ (solid).

This is due to the fact that there exist dynamical flavor branes which intersect the probe brane corresponding to the test quarks. The open string stretching between the probe and those dynamical branes has nearly massless modes at the intersection. In fact their mass is expected [30] to scale as $M_Q/\sqrt{\lambda}$ (where λ denotes the bare 't Hooft coupling), which is parametrically smaller than M_Q since $\lambda \gg 1$ in the supergravity regime. The decay into a pair of mesons of this kind can happen when the energy of the static configuration, $2M_Q + V(L)$, equals their total mass, $2M_Q/\sqrt{\lambda}$. Hence

$$V(L_s) = -2M_Q \left(1 - \frac{1}{\sqrt{\lambda}} \right). \quad (7.2)$$

Due to the presence of the Landau pole, M_Q cannot be as large as we want and it can be interesting to study how L_s varies with the flavor parameters. Relevant plots (with $\sqrt{\lambda} = 50$) can be found in figure 4. As expected, the screening length decreases as N_f increases and it is an increasing function of m_q . From the Coulomb-like behavior of $V(L)$ we expect, moreover, that L_s is decreasing with M_Q . The fact that the potential also depends on the cutoff M_Q (at fixed L , $|V(L)|$ is an increasing function of M_Q as we have seen) does not alter this behavior, as it is evident in the figure.

A recent proposal for improving the quenched approximation in QCD so to take into account the screening effects of dynamical flavors can be found in [31].

7.2 The string breaking length

It is important to notice that due to the smearing, the decay of the static configuration into a given pair of heavy-light bound states is suppressed by $1/N_c$ and not by N_f/N_c as it would happen in a setup with localized parallel flavor branes. A decay rate which is only N_f/N_c suppressed can be obtained by considering the possibility of decaying into an arbitrary sizable fraction of the N_f types of heavy-light bound states. We call the minimal $\bar{Q}Q$ separation at which this kind of decay can happen, the “string-breaking length” L_{sb} . We choose to define L_{sb} as the length at which a dynamical quark-antiquark pair with the same internal charges as the test quarks can be popped out from the vacuum. The corresponding produced heavy-light bound states have a mass given by $M_Q - m_q$ and thus a large binding energy $E_b = -2m_q$ independent on the 't Hooft coupling. The open string describing these “mesons” is just stretched along the radial direction of the geometry (5.1) between the probe brane at ρ_Q and a parallel dynamical flavor brane at ρ_q [32]. Since the

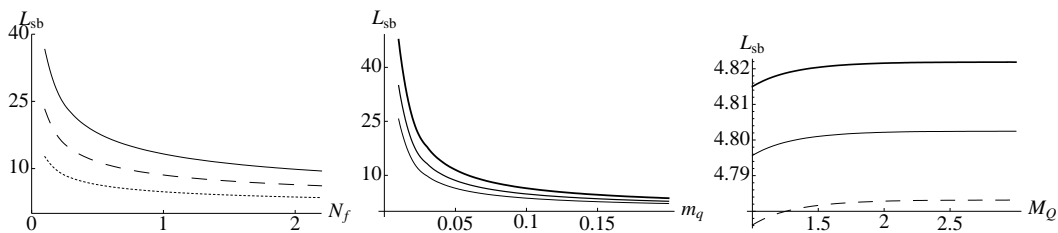


Figure 5: From left to right, the string breaking length L_{sb} as a function of: N_f , for $m_q = 0.03, 0.05, 0.1$, top to bottom ($M_Q = 2$); m_q , for $N_f = 0.5, 1, 2$, top to bottom ($M_Q = 2$); M_Q for $N_f = 1, m_q = 0.1$ (thick), $N_f = 1.01, m_q = 0.1$ (solid), $N_f = 1, m_q = 0.101$ (dashed).

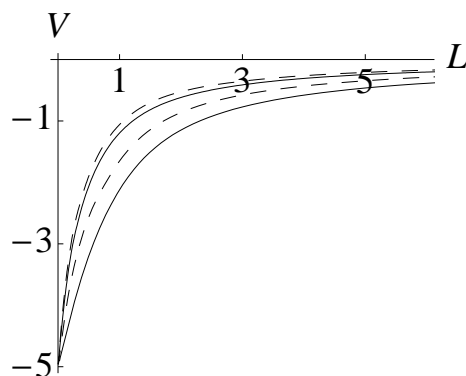


Figure 6: The potential $V(L)$ in the step approximation (dashed lines) and with the correct $N_f(\rho)$ function (solid lines), for $M_Q = 2.5$, $N_f = 1$ and $m_q = 0.3, 1.5$ (top to bottom).

interactions between the two produced “mesons” is negligible, it follows from our definition that L_{sb} is just the solution of

$$V(L_{sb}) + 2M_Q = 2(M_Q - m_q) \quad \Rightarrow \quad V(L_{sb}) = -2m_q. \quad (7.3)$$

Figure 5 shows that L_{sb} is a decreasing function of both N_f and m_q . The latter behavior is due to the shift in the meson mass as we vary m_q . L_{sb} is an increasing function of M_Q which tends to flatten at large M_Q .

7.3 Comparison with the Heaviside approximation

It is instructive to compare the behavior of the Wilson loop observables as obtained using the correct string dual solution with $N_f(\rho)$ given in (4.18), with those obtained using the Heaviside step function approximation. In figure 6 we compare the plots for $V(L)$ at different values of the parameters. The figure confirms that the step approximation, in the flavored KW model, works quite well when m_q is small, or, better said, when $M_Q - m_q$ is large. Moreover, it is possible to show that the approximation is even better if the step is placed where the function $N_f(\rho)$ has reached half of its value $N_f(\rho_{step}) = N_f/2$.

7.4 Comparison with field theory expectations

The behavior of the potential $V(L)$ as well as that of the critical lengths L_s and L_{sb} , as a

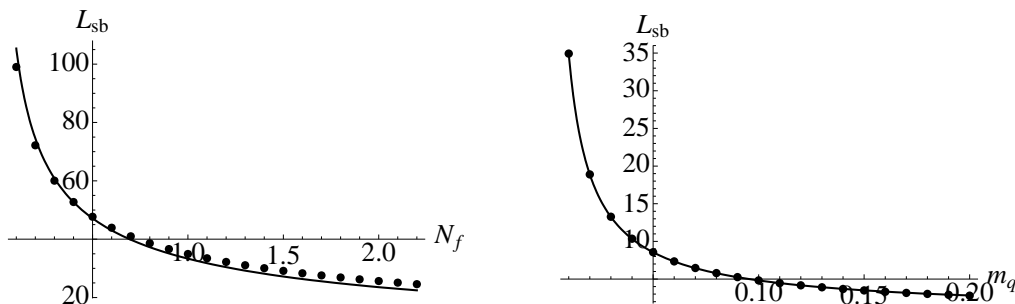


Figure 7: Fits of the numerical data (dots) with an analytic formula like (7.7); on the left, $m_q = 0.01, M_Q = 2$; on the right, $N_f = 1, M_Q = 2$.

function of the flavor parameters, fits very well with a simple analytical result suggested by field theory and AdS/CFT arguments.

We know that in the low energy regime, when the massive flavors are integrated out, our model has vanishing beta function. In the string dual description this is reflected into the constancy of the dilaton $\phi = \phi_{IR}$ in the $\rho < \rho_q$ region. The addition of flavors gives a negative beta function, whose one loop coefficient behaves as $b_{UV} = -\frac{3}{4}N_f$. In the perturbative regime $m_q \ll \Lambda_{UV}$ (where Λ_{UV} indicates the position of the Landau pole), we can match the UV and IR couplings at the scale m_q so to get the IR value of the coupling

$$\frac{8\pi^2}{g_{IR}^2} = \frac{3}{4}N_f \log \frac{\Lambda_{UV}}{m_q}. \quad (7.4)$$

This expression is nothing but the field theory rewriting of the stringy result we have found for the dilaton in the step function approximation, $e^{\phi_{IR, st}} = -4\pi/(3N_f\rho_q)$. Indeed, when $\rho_q \rightarrow -\infty$ we can safely identify $\rho_q \approx \log(m_q/\Lambda_{UV})$. Now, remembering the expression for the $\bar{Q}Q$ potential of a conformal theory with an AdS dual [28], we expect that for our model, in the strong coupling regime

$$V_{\bar{Q}Q}(L) \approx -\frac{\sqrt{g_{IR}^2 N_c}}{L} \approx -\frac{\sqrt{N_c}}{\sqrt{N_f \log \frac{\Lambda_{UV}}{m_q}}} \frac{1}{L}, \quad (7.5)$$

in the large L limit. From this expression we can deduce, in the same limits, the screening length L_s

$$L_s \approx \frac{\sqrt{N_c}}{M_Q \sqrt{N_f \log \frac{\Lambda_{UV}}{m_q}}}, \quad (7.6)$$

and the string breaking length L_{sb}

$$L_{sb} \approx \frac{\sqrt{N_c}}{m_q \sqrt{N_f \log \frac{\Lambda_{UV}}{m_q}}}. \quad (7.7)$$

The above analytical functions share with our numerical results the same qualitative behavior with the flavor parameters. The expression for the screening length is not expected

to fit well with our data, since the approximation we used to get eq. (7.6) requires L to be large. Instead, in figure 7 we have fitted our numerical data with a formula for L_{sb} as in eq. (7.7) - treating Λ_{UV} and an overall constant as parameters - finding a very good agreement.

Acknowledgments

We are grateful to Carlos Núñez for his crucial observations and his participation in the early stages of this project. We would also like to thank D. Areán, R. Argurio, S. D. Avramis, F. Benini, R. Casero, S. Cremonesi, N. Drukker, E. Imeroni, S. Kuperstein, A.V. Ramallo, K. Sfetsos, M. Teper, W. Troost. This work has been supported by the European Commission FP6 programme MRTN-CT-2004 v-005104, “Constituents, fundamental forces and symmetries in the universe”. F. B. is also supported by the Belgian Fonds de la Recherche Fondamentale Collective (grant 2.4655.07), by the Belgian Institut Interuniversitaire des Sciences Nucléaires (grant 4.4505.86) and the Interuniversity Attraction Poles Programme (Belgian Science Policy). A. C. is also supported by the FWO - Vlaanderen, project G.0235.05 and by the Federal Office for Scientific, Technical and Cultural Affairs through the Interuniversity Attraction Poles Programme Belgian Science Policy P6/11-P. A. P. is also supported by a NWO VIDI grant 016.069.313 and by INTAS contract 03-51-6346. F. B. and A. C. thank the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support during the completion of this work.

A. D7 embeddings from the orbifold

There are two main classes of holomorphic embeddings for D7 branes on the conifold: class I has $z_1 = \mu$ [22] as representative embedding equation, while class II is represented by $z_1 - z_2 = \mu$ [23]. The two classes are argued to correspond to two different ways of adding flavors to the KW model. Let us focus here on the massless $\mu = 0$ case. Class I embeddings have two branches of D7 branes, each branch adding fundamental matter to the first node of the KW quiver and antifundamental matter to the second. In the class II case a D7-brane provides fundamental and antifundamental matter to one node. Since the KW theory can be obtained starting from the $\mathcal{N} = 2$ conformal theory describing N_c D3-branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity [10] it could be useful to classify the possible kinds of D7 embeddings starting from the orbifold picture. This could also help in finding the correct superpotential terms corresponding to the chosen embeddings. A discussion on these issues can be found in [22].

The orbifold theory is a quiver with gauge group $SU(N_c) \times SU(N_c)$ and, in terms of $\mathcal{N} = 1$ components, the same bifundamental multiplets A_i, B_i as the KW theory, plus two adjoint supermultiplets Φ_1 and Φ_2 , one for each of the gauge groups. The superpotential of the theory schematically reads

$$W_{\text{orb}} = \Phi_1(A_1 B_1 - A_2 B_2) - \Phi_2(B_1 A_1 - B_2 A_2). \tag{A.1}$$

There are mainly two ways of adding massless flavors to both nodes of the quiver: either we add to the above superpotential cubic terms coupling the new flavors to the bifundamentals, i.e. (modulo $SU(2) \times SU(2)$ rotations)

$$W_{\text{orbI}} = W_{\text{orb}} + \tilde{q}_1(A_1 - A_2)q_2 + \tilde{q}_2(B_2 - B_1)q_1, \quad (\text{A.2})$$

or we couple the new flavors with the adjoint supermultiplets

$$W_{\text{orbII}} = W_{\text{orb}} + \tilde{q}_1\Phi_1q_1 - \tilde{q}_2\Phi_2q_2. \quad (\text{A.3})$$

These two possibilities have been considered in the T-dual Type IIA picture [33, 34] to which we refer for the sign conventions. The superpotential in (A.2) reduces supersymmetry by one half, while that in (A.3) does not break the $\mathcal{N} = 2$ supersymmetry (the corresponding string model is just an orbifold of the D3D7 system in flat space). In case I each branch of D7 branes extends only along two directions inside the orbifold. In case II, the D7-branes are taken to be extended along the 4 orbifolded directions. It is easy, in this case, to see that the flavored theory has a Landau pole: the one loop coefficient of the beta function (which does not get perturbative corrections beyond one-loop due to the $\mathcal{N} = 2$ supersymmetry) for each group is equal to $b = 3N_c - N_c - 2N_c - N_f = -N_f$ (where the negative terms $-N_c, -2N_c, -N_f$ are the contributions of the adjoint, bifundamental and fundamental matter superfields, respectively) and on each node the theory is thus IR free.

Let us now go to the $\mathcal{N} = 1$ conifold theory by adding to the above superpotentials the appropriate mass term for the adjoints

$$W_m = \frac{m_\Phi}{2}(\Phi_1^2 - \Phi_2^2). \quad (\text{A.4})$$

Integrating out the adjoint fields in case I, will produce a superpotential of the form

$$W_I = W_{\text{KW}} + h_1\tilde{q}_1(A_1 - A_2)q_2 + h_2\tilde{q}_2(B_2 - B_1)q_1, \quad (\text{A.5})$$

while in case II one will get

$$W_{\text{II}} = W_{\text{KW}} + \hat{h}_1\tilde{q}_1[A_1B_1 - A_2B_2]q_1 + \hat{h}_2\tilde{q}_2[B_1A_1 - B_2A_2]q_2 + k_1(\tilde{q}_1q_1)^2 + k_2(\tilde{q}_2q_2)^2. \quad (\text{A.6})$$

Writing the effective mass terms for the flavors in a compact matrix form $\tilde{q}Mq$, the equation $\det M = 0$ gives, for the case I, $A_1B_1 + A_2B_2 - A_1B_2 - A_2B_1 = 0$, i.e. $z_1 + z_2 - z_3 - z_4 = 0$. This is nothing but one of the possible rotations of the $z_1 = 0$ embedding: in fact it is our generalized embedding (3.11) with $\alpha_3 = \alpha_4 = -1$. In case II, $\det M = 0$ gives $A_1B_1 - A_2B_2 = 0$, i.e. the $z_1 - z_2 = 0$ embedding equation.

The massive generalizations of the embeddings above correspond to the addition of standard mass terms $m_1\tilde{q}_1q_1 + m_2\tilde{q}_2q_2$ to the massless superpotential. In case I the mass terms break the classical flavor symmetry group to $U(N_f)$, while in case II, the $U(N_f) \times U(N_f)$ symmetry is preserved.

It is also tempting to write the class I D7 embeddings using the orbifold coordinates. The $\mathbb{C}^2/\mathbb{Z}_2$ orbifold is described by the $xy = z^2$ equation in \mathbb{C}^3 . The massless embeddings of class I, should thus be described by an equation like $z = 0$, giving rise to the two branches $x = 0$ and $y = 0$ corresponding to D7 branes extended only over one half of the orbifold directions.

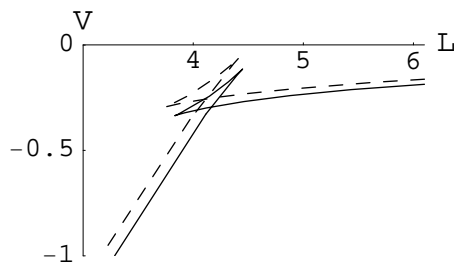


Figure 8: Plots of $V(L)$ with $c_2 = 10$, $m_q = 0.2$, $N_f = 1$, $M_Q = 2.5$. The dashed (resp. solid) line refers to the result obtained using the Heaviside approximation (resp. the correct solution with $N_f(\rho)$).

B. Dependence on the integration constants

In section 5 we have constructed the background solution obtained with a precise prescription for the integration constants \tilde{c}_1, c_2, c_3 . We would like to briefly discuss here what would happen with a different choice of constants. As explained in section 5.4, $\tilde{c}_1 \neq 0$ would produce a singularity in the IR which we want to avoid. On the other hand, varying c_3 just produces a rescaling of e^f, e^g, h and, for the static quark-antiquark system, a rescaling of L and V which does not change the qualitative form of the $V(L)$ curves. Finally, $c_2 < 0$ would make the geometry singular at some point below the Landau pole, a behavior that we also want to avoid. Let us thus focus on modifying our original prescription $c_2 = 0$ to $c_2 > 0$. The behavior of the $V(L)$ static potential is crucially affected by the value of c_2 , as shown in figure 8. For every $c_2 > 0$ the large L ($\rho_0 \rightarrow -\infty$) regime of $V(L)$ is unchanged: the potential approaches zero from below since the theory is IR conformal. However, above a certain critical value of c_2 (which depends on the other physical parameters), one observes a double turnaround of the $V(L)$ graph at shorter distances: the potential has a discontinuity in the first derivative, and there is a first order phase transition.

This kind of behavior was first found in [35] and afterwards discussed in several different contexts [36, 7]. In the region of L where $V(L)$ is triple-valued, the point with lowest V is stable, the intermediate one is metastable and the one with larger $V(L)$ is perturbatively unstable (always referring to longitudinal perturbations of the string). Stability issues of this kind of strings were discussed in detail in [37].

The double turnaround region is visible also in the massless $m_q = 0$ case. As it is evident from figure 8, the Heaviside approximation does not modify qualitatively the double turnaround behavior of the potential.

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